Sources of Error in Obtaining Damping Properties of Hard Coatings from Vibration Tests

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Thin coatings of high damping capacity are often added to structures to reduce the amplitudes of resonant vibration. The properties of such materials are typically determined by applying thin coatings to specimens, testing the coated object, and deducing coating properties form the response of the coated system. Several sources of error or uncertainty limit the accuracy to which desired properties are obtained. Sources of such errors are considered and a methodology is developed for predicting the uncertainty in the desired properties from the uncertainties in test measurements and parameters. Examples for fully and partially coated beams are given.

INTRODUCTION

In order to design structural components incorporating a free layer coating added for the purpose of reducing vibratory amplitudes, it is necessary that the material properties of the coating (storage modulus and loss modulus, or equivalently, storage modulus and loss factor) be known. The physical characteristics of materials suitable for use as dissipative coatings are generally such that the desired tests of homogeneous samples under homogeneous states of strain (stress) are not feasible. As an alternative, the practice of coating a test specimen of relatively simple geometry and then measuring the system response of the specimen-coating system has been commonly adopted. One can then quantify the contribution of the coating to the system level response, and from that, deduce the material properties of the coating material.

Two test configurations have been employed. In the first, a beam is fully covered on both sides with the coating material. This configuration, as applied to a cantilever beam, is shown in Fig. 1, and is suitable for testing in any of several vibratory modes. This is a well established technique for the determination of the properties of linear materials. However, for a non-linear material (i.e., with material loss factor a function of amplitude), account must be taken of the fact that the strain (stress) distribution varies significantly (i.e. 100%) over the length of the beam. A test configuration employing a partially coated beam, also shown in Fig. 1, has been suggested as a means of avoiding large variations in strain within the coating material.

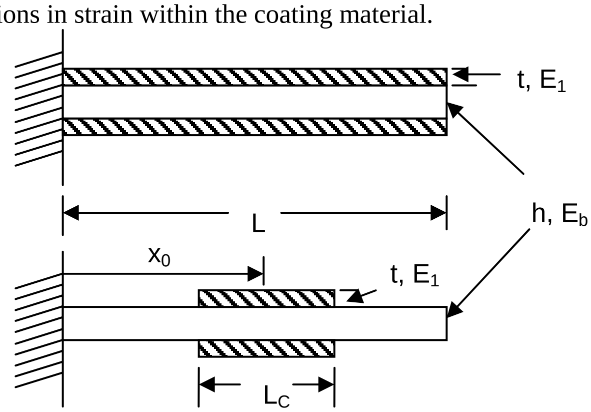

Figure 1. Two Configurations for Evaluation the Damping Properties of Coatings

By minimizing the length of the coated section, L_C, and locating it at a region of maximum strain, the lengthwise variations in strain can be minimized. The disadvantages of this method are primarily two. First, the requirement to obtain system loss factors sufficiently high as to enable meaningful measurements necessitates the use of coatings of substantial thickness, leading to significant variations of strain through the thickness of the coating. A second disadvantage is that the requirement to locate the coating in a region of maximum strain means that the specimen must be designed for a specific mode (natural frequency) of the test specimen.

We assume that the coating modulus may be adequately described by a one-dimensional complex modulus [1]:

$$E^* = E_1 + jE_2 = E_1(1 + j\eta) \tag{1}$$

where E_1 is a real valued storage modulus, equivalent to the Young's modulus; E_2 is a real valued loss modulus; and the material loss factor, η, is the ratio of the two. If the material is linear and the amplitude, ε (x.y,.z), of a fully reversed cyclic strain distribution is known, the loss modulus is related [2] to the energy dissipated in one complete cycle of oscillation by

$$D = \pi E_2 \int_V \varepsilon^2 dV \tag{2}$$

and the storage modulus is related to the peak energy stored during the cycle by

$$U = \frac{E_1}{2} \int_V \varepsilon^2 dV \tag{3}$$

The loss factor, as defined in Eq. (1), then becomes

$$\eta = \frac{E_2}{E_1} = \frac{D/(\pi \int_V \varepsilon^2 dV)}{2U / \int_V \varepsilon^2 dV} = \frac{D}{2\pi U} \tag{4}$$

and may also be interpreted as the energy dissipated per radian of oscillation, normalized by the peak energy stored. While the loss factor is of interest, It is the loss modulus that is of primary significance since, as is seen from Eq. (2), it is the quantity that determines the absolute rate at which energy is dissipated.

When the loss and storage modulii are determined from the system response, it is presumed that all measurements and system parameters are precisely known quantities. In general, this is not the case. Rather, each is only known to within some degree of uncertainty. It is the objective of this work to evaluate the influence of such possible errors on the accuracy of material modulii as deduced from measurements on coated structures. But the quantification of the influence of these possible errors requires knowledge of the functional dependence of the desired properties on each of these parameters and measurements, Thus, we must begin with an analysis of the coated systems in sufficient detail as to establish the relationship between the system response and quantities upon which that response depends.

IDENTIFICATION OF MEASUREMENTS AND PARAMETERS

Determinng the Loss Modulus In order to determine the loss modulus of a coating material from system level measurements of the loss factors of the coated and uncoated systems, it is first necessary to isolate the contribution of the dissipation in the coating to the system loss factor. In general, the system loss factor for a coated system may be written as:

$$\eta_{SYS} = \frac{1}{2\pi} \frac{D_{Diss} + D_{Extraneous}}{U_{Diss} + U_{Non-Diss}} \tag{5}$$

where D_{Diss} is the energy dissipated by the damping coating, $D_{Extraneous}$ is the energy dissipated by the substrate, grips and losses to the environment, U_{Diss} is the energy stored in the damping coating, $U_{Non\text{-}Diss}$ is the energy stored in the substrate and grips, and η_{SYS} is the system loss factor, a consequence of dissipation due to all sources.

The extraneous losses may be evaluated if, prior to application of the coating, the bare specimen is tested under the same conditions (e. g. temperature, grip configuration, and surrounding medium) as will be used in the testing of the coated beam. If the resulting loss factor is found to be amplitude dependent, then it should be insured that that the bare beam testing is done at amplitudes comparable to those to be experienced with the coated specimens. The system loss factor will be:

$$\eta_{Bare} = \frac{1}{2\pi}\frac{D_{Extraneous}}{U_{Non-Diss}} \tag{6}$$

We may use this to isolate the contribution of the coating to the system loss factor. Let

$$\eta_{SYS-Coat} = \frac{1}{2\pi}\frac{D_{Diss}/U_{Diss}}{(1+1/R_{SE})} = \eta_{SYS} - \eta_{BARE}\frac{1}{1+R_{SE}} \tag{7}$$

where the strain energy ratio is defined as:

$$R_{SE} = U_{Diss}/U_{Non-Diss} \tag{8}$$

The contribution of the coating to the system loss factor can then be evaluated in terms of the observed loss factor of the coated beam (η_{SYS}) and the observed loss factor of the bare beam (η_{BARE}), and the strain energy ratioNote that the strain energy ratio used here differs from that used in the Method of Modal Strain Energy. In that case, the denominator is taken as the sum of the two energies so that $R_{MSE} = R_{SE} / (1+ R_{SE})$.

The contribution of the coating to the system loss factor, Eq, (7), may prove to be dependent of the strain amplitude. This is normally indicative of the presence of a non-linear damping mechanism. However, the contribution may be related to material parameters of the coating material. For a linear material, the loss modulus is related to the energy dissipated by Eq. (2). For materials with nonlinear damping properties, an empirical form

$$D = J\varepsilon^{N} \tag{9}$$

introduced by Lazan [3] is often useful. Here, D is the energy dissipated in a unit volume at uniform strain during one complete cycle of fully reversed strain of amplitude ε. The energy dissipated in a volume V is then the integral. This may be expressed in terms of a strain-dependent effective loss modulus, $\overline{E}_{2,}$ defined in such a way as to lead to the observed energy dissipated per cycle, i.e. ,

$$D_{DISS} \equiv \int_{Vol-Coat} J\varepsilon^{N} dV \cong \pi\overline{E}_2(\varepsilon_{MAX}) \int_{Vol-Coat} \varepsilon^2 dV \tag{10}$$

The energy stored in the coating is given by Eq. (3). While strictly true only for a linear elastic material, its may be approximated for a weakly nonlinear material by using an amplitude dependent storage modulus, $\overline{E}_1$, analogous to the amplitude dependent loss modulus. Substitution of Eq. (3 and 10 into Eq. (11) then gives

$$\eta_{SYS-Coat} = \frac{\overline{E}_2(\varepsilon_{MAX})/E_1}{(1+1/R_{SE})} \tag{11}$$

which may be solved for the effective loss modulus. For simplicity, we take here the uniaxial storage modulus E_1 to be constant. Note that the numerator is the material loss factor. Eq. (7) may then be used to write

$$\overline{E}_2(\varepsilon_{MAX}) = E_1[(1+R_{SE})\eta_{SYS} - \eta_{BARE}]/R_{SE} \tag{12}$$

Thus, if the storage modulus of the coating material, E_1, and the strain energy ratio, R_{SE}, are known, the coating loss modulus can be determined from the measurements of loss factors for the coated (η_{SYS}) and uncoated (η_{BARE}) systems.

The modulus, $\overline{E}_2$ (ε_{MAX}) so determined is not truly a material property, as it is dependent on the details of the strain distribution. The true amplitude dependent modulus, E(ε) of a non-linear material may be found, however, from the solution of the integral equation

$$\int_{Vol} E_2(\varepsilon)\varepsilon^2 dV = \overline{E}_2(\varepsilon_{MAX})\int_{Vol}\varepsilon^2 dV \tag{13}$$

for various values of ε_{MAX}, using values of E_2 (ε_{MAX}) obtained from measured loss factors by the use of Eq. (12). This is readily accomplished [4] if the non linear damping is adequately described by Eq. (9).

The Strain Energy Ratio The strain energy ratio, as defined in Eq. (8), requires the evaluation of the stored (elastic) energy in the dissipative (coating) and non dissipative (substrate) components of the system. Assuming that the coating is perfectly bonded to the beam, the strain distribution is continuous through the thickness of beam and coating and varies linearly with distance from the mid-plane according to:

$$\varepsilon(x,z) = Cz\{d^2\hat{X}_n(x)/dx^2\} \tag{14}$$

Here, z is the distance from the mid-plane, $\hat{X}_n$ is the mode shape of the deflected system in the nth mode, and C is an amplitude. If the substrate thickness is h, the strain at the beam-coating interface, e(x) = ε(x,h/2) is readily determined from the local curvature. For the fully covered beam, the mode shape is an eigenfunction of a uniform Bernoulli-Euler beam, i.e. $\hat{X}_n(x) \equiv X_n(x)$, but for the partially covered beam it may be expected to differ somewhat.

If the substrate beam has uniform values of modulus E, width W, and thickness h, the strain energy is:

$$U_{Non-Diss} = \frac{E}{2}\int_{Vol-Beam}\varepsilon(x)^2 dV = \frac{C^2}{2}\frac{Eh^3W}{12}\int_0^L\{\frac{d^2\hat{X}_n(x)}{dx^2}\}^2 dx \tag{15}$$

The strain energy in a coating of uniform thickness t fully covering both sides of a beam of width W between x = L_1 and L_2 is

$$U_{Diss} = \frac{E_1}{2}\int_{Vol-Coat}\varepsilon^2 dV = \frac{E_1C^2}{2}2W(\frac{h}{2})^2 t\Theta(t/h)\int_{L_1}^{L_2}[\frac{d^2\hat{X}_n(x)}{dx^2}]^2 dx \tag{16}$$

where the thickness integral in expressed in terms of a dimensionless function Θ(t/h) defined by:

$$\Theta(t/h) = [1 + (2t/h) + (2t/h)^2/3] \tag{17}$$

For small values of t/h, Θ(t/h) ⇒ 1, but the quantity increases significantly as the thickness ratio is increased. For example, Θ(0.1)= 1.213, Θ(0.2) = 1.453 and Θ(0.3) = 1.720.

The strain energy ratio is then found from Eq. (15) and Eq. (16) to be:

$$R_{SE} = \frac{U_{Diss}}{U_{Non-Diss}} = 6\frac{\overline{E}_1 t}{Eh}\Theta(t/h)\int_{L_1}^{L_2}[\frac{d^2\hat{X}_n(x)}{dx^2}]^2 dx \Big/ \int_0^L\{\frac{d^2\hat{X}_n(x)}{dx^2}\}^2 dx \tag{18}$$

A small amount of energy stored in the grips may be accounted for by replacing E with an effective substrate modulus, $\overline{E}$, determined as being the value necessary to predict a measured bare beam natural frequency.

Because the strain distribution in both the coating and the beam is different in the case of fully covered and partially covered beams, we must evaluate separately the strain energy ratio for each of the two cases.

In the case of the fully covered beam, the integrals in the numerator and denominator of Eq, (18) are the same, and the strain energy ratio is simply the first group of terms. For the partially coated beam, we assume the coated region to be short, and, since a high strain in the coating is desired, that the coated region is located at one of the interior maxima in strain. In consequence, the strain at the interface of the coated region is nearly uniform at e_0 , a fraction, λ, of the maximum strain at the root, ε_{MAX}. For the second bending mode, this ratio is $\lambda = 0.720$ at $x_o/L = 0.53$; for the third, $\lambda = 0.822$ at $x_o/L = 0.71$; and for the fourth, $\lambda = 0.849$ at $x_o/L = 0.79$.

Unless the storage modulus of the coating is extremely low, the increased stiffness in the coated region will influence the mode shape. The stiffening influence of the coating may be estimated by noting that the beam moment must be continuous across the ends of the coated region. In consequence, by equating the moment in the coated region to that at the adjacent uncoated region, we may write that

$$(EI)_{EFF} \frac{d^2w}{dx^2}\Big|_{Coated} = EI_{Beam} \frac{d^2w}{dx^2}\Big|_{Bare} \tag{19}$$

The reduction in interface strain may then be estimated from the effective stiffness of the coated region. Let

$$\alpha = \frac{e_C}{e_0} = \frac{(h/2)d^2w/dx^2\ |_{Coated}}{(h/2)d^2w/dx^2\ |_{Bare}} = [1 + 6\frac{E_1 t}{E_b h}\Theta(\frac{t}{h})]^{-1} \tag{20}$$

where e_C is the interface strain <u>within</u> the coated region, and e_o is the surface strain just <u>outside</u>.

If the curvature is unaffected outside of the coated region, and the interface strain is taken as uniform within, the strain in the non-dissipative substrate under the coating can be related to that of the uncoated beam The fractional reduction in the strain energy of the substrate can then be evaluated and becomes:

$$\delta_U = 4\lambda^2(1-\alpha^2)(L_C/L) \tag{21}$$

so that the strain energy ratio for the partially coated beam is

$$R_{SE-P} = 6\frac{\bar{E}_1 t}{Eh}\Theta(\frac{t}{h})\frac{4\lambda^2\alpha^2 L_C/L}{1-\delta_U} \tag{22}$$

Note that the strain energy ratio for the fully coated beam ($L_C/L = 1$) results from setting the strain ratio λ to the rms value of ½ and setting $\alpha = 1$so that $\delta_U = 0$.

<u>Determining the Storage Modulus</u> The determination of the loss modulus from Eq. (12) was found to require knowledge of the storage modulus, both explicitly and implicitly through the strain energy ratio, Eq. (22). In general, the storage modulus will not be known *a priori*, and must be determined by experiment. This may be done by measurements of the frequency of coated and uncoated beams. We may estimate the frequency of the coated beam from a Rayleigh quotient, with the strain energies in the numerator being those found above.

$$\omega^2 = \frac{U_{Non-Diss} + U_{Diss}}{T_{Diss}/\omega^2 + T_{Non-Diss}/\omega^2} = \frac{(1+R_{SE})U_{Non-Diss}}{T_{Diss}/\omega^2 + T_{Non-Diss}/\omega^2} \tag{23}$$

In the case of the partially coated beam, we assume that, since the influence of the coating is to reduce the local strain by a factor 1-α that a reduction in beam displacement also occurs, leading to a reduction in kinetic energy of the system proportional to the square. As there does not seem to be a means of evaluating this reduction by elementary means, we adopt a semi-empirical form

$$\delta_T = G(1-\alpha)^2 \tag{24}$$

wth the geometrical factor, G, left to be determined by some other means. Thus, we take

$$T_{Non-Diss} = (1-\delta_T)\omega^2 WhL\rho_b \frac{C^2}{2}\frac{1}{L}\int_0^L [X_n(x)]^2\,dx \tag{25}$$

for the substrate, where X_n is taken as the eigenfunction of the beam. For a short coated region, centered at a point of maximum strain, x_0, we take displacement of the coating to have a uniform value $X_n(x_0)$. The kinetic energy of the coating on both sides of the beam then becomes:

$$T_{Diss} \cong \omega^2 Wt\rho_c(1-\delta_T)C^2 L_C[X_n(x_0)]^2 \tag{26}$$

so that the kinetic energy ratio for the partially coated beam is:

$$R_{KE-P} = \frac{2t\rho_c L_C[X_n(x_0)]^2}{hL\rho_b \frac{1}{L}\int_0^L [X_n(x)]^2\,dx} = \frac{8t\rho_c L_C}{hL\rho_b}\left[\frac{X_n(x_0)}{X_n(L)}\right]^2 \tag{27}$$

since the integral of the squared normed eigenvalue for any mode of a cantilever beam has a value of ¼. Taking into account the reduction in strain energy due to the coating, Eq. (21), and the reduction in kinetic energy, Eq. (24), the Rayleigh quotient of Eq. (23) becomes:

$$\omega^2 = \frac{(1-\delta_U)(1+R_{SE-P})}{(1-\delta_T)(1+R_{KE-P})}\frac{\frac{Eh^3}{12}\int_{L_1}^{L_2}\left\{\frac{d^2X_n(x)}{dx^2}\right\}^2 dx}{h\rho_b\int_0^L [X_n(x)]^2\,dx} \tag{28}$$

But the fraction on the right is the frequency of the uncoated beam in the n^{th} mode. Thus, the natural frequencies of the partially coated beam may be related[1] to those of the uncoated beam through

$$\frac{\omega^2}{\omega_0^2}\Big|_{P-n} = \frac{(1-\delta_U)(1+R_{SE-P})}{(1-\delta_T)(1+R_{KE-P})} \tag{29}$$

with the necessary parameters, δ_U, R_{SE-P}, and R_{KE-P}, being as given by Eqs. (21, 22, and 27). In the case of the partially coated beam, each of these quantities is dependent on the mode number. For the fully covered beam, the kinetic energy ratio is found from Eq. (27) by setting $L_C/L = 1$ and the displacement ratio to the rms value of ½.

Thus, measurements of the natural frequencies of the uncoated and coated beams in the same mode provide a means of evaluating the strain energy ratio and, from that, the storage modulus. In the case of the fully coated beam this is straightforward. In the case of the partially coated beam, the presence of the storage modulus in the parameter α, Eq. (20) as well as in the strain energy ratio, R_{SE} , Eq. (22) makes this somewhat more difficult.

THE ESTIMATION OF UNCERTAINTY

For both the fully and partially coated beams, the loss modulus of the coating was found to depend on the storage modulus of the coating, the strain energy ratio, and the measured values of the system loss factor for both the bare and coated beams. The dependence is given by Eq. (12). It is useful to normalize the storage and loss modulii of the coating by the modulus of the substrate and then introduce two stiffness parameters,

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[1] A comparison of this frequency with results from a finite element analysis [5] gave agreement within 1% with G = 1 for an example with $L_C/L = 1.18$ and $\alpha = 0.442$. Other values of these parameters have not been considered.

$$S_1 = \{\bar{E}_1(\varepsilon_{MAX})/E\}(1+R_{SE})/R_{SE} \tag{30}$$

$$S_2 = \{\bar{E}_1(\varepsilon_{MAX})/E\}/R_{SE} \tag{31}$$

Eq. (12) then becomes

$$\bar{E}_2(\varepsilon_{MAX})/E = S_1\eta_{SYS} - S_2\eta_{BARE} \tag{32}$$

From this, it is seen that that uncertainties in the determination of the loss modulus will arise from four sources: uncertainties in the two measured loss factors, and uncertainties in the two stiffness parameters.

We consider first the influence of uncertainty in the measurement of loss factors. If the fractional uncertainties in the evaluation of the loss factors of bare and coated beams can be estimated as

$$f_S = \frac{|\Delta\eta_{SYS}|}{\eta_{SYS}} \qquad f_B = \frac{|\Delta\eta_{BARE}|}{\eta_{BARE}} \tag{33a,b}$$

then the the resulting uncertainty in the estimation of the loss modulus is

$$\Delta\{\bar{E}_2(\varepsilon_{MAX})/E\}|_\eta = S_1\eta_{SYS}\, f_S + S_2\eta_{BARE}\, f_B \tag{34}$$

Since the loss factor measurements are independent quantities, the total error is taken as the sum of the absolute values of the two contributions. This is a somewhat conservative assumption, as certain systematic errors in the measurement of loss factors could lead to both being over, or under, predicted.

Because the loss modulus also depends on the stiffness parameters S_1 and S_2, and because these depend on other system parameters that can only be measured with some degree of uncertainty, an additional uncertainty will arise in the determination of the loss modulus,

$$\Delta\{\bar{E}_2(\varepsilon_{MAX})/E\}|_S = \eta_{SYS}\Delta S_1 - \eta_{BARE}\Delta S_2 \tag{35}$$

In this case, the quantities S_1 and S_2 can not be expected to vary independently. In consequence, the dependence of each on the independent test parameters must be determined and the uncertainties combined before the absolute values are taken.

For the purpose of this analysis, we will take the length of the beam and the length and location of the coated region to be given, and the coating width to be precisely the beam width, so that these do not contribute to uncertainties in the determination of the modulus. Thus, the stiffness parameters S_1 and S_2 are taken to be explicit functions only of the thickness ratio, t/h and the modulus ratio E_1/E. The latter ratio, however, is not generally known, but must be inferred from measured values of the natural frequency, a function of the three ratios E_1/E, t/h, and the density ratio, ρ_C/ρ_B. Thus, by formulating the stiffness ratios S_1 and S_2 as functions of three independent variables: the thickness ratio, the density ratio, and the ratio of the coated to uncoated beam frequencies, we may determine the uncertainty in the determination of the stiffness ratios in terms of the uncertainties in these measurable quantities.

For fully and partially covered beams, the storage modulus is related to the frequency of the nth mode through the frequency ratio of Eq. (29) with the strain energy given by Eq. (22), the kinetic energy ratio by Eq. (27) and the parameters δ_U and δ_T given by Eqs. (21) and (24). The stiffness parameters defined by Eqs. (30) and (31) may also be expressed in terms of these parameters so as to enable the formation of an error array, Δ_{IJ}, formed by taking the appropriate partial derivatives. In the case of the fully covered beams the parameter α must be set to unity before the partial derivatives are evaluated.

$$\begin{bmatrix} \Delta(\frac{\omega}{\omega_0})^2 \\ \Delta(S_1) \\ \Delta(S_2) \end{bmatrix} = \begin{bmatrix} \frac{\partial(\omega/\omega_0)^2}{\partial(E_1/E)} & \frac{\partial(\omega/\omega_0)^2}{\partial(\rho_C/\rho_B)} & \frac{\partial(\omega/\omega_0)^2}{\partial(t/h)} \\ \frac{\Delta S_1}{\partial(E_1/E)} & \frac{\Delta S_1}{\partial(\rho_C/\rho_B)} & \frac{\Delta S_1}{\partial(t/h)} \\ \frac{\Delta S_2}{\partial(E_1/E)} & \frac{\Delta S_2}{\partial(\rho_C/\rho_B)} & \frac{\Delta S_2}{\partial(t/h)} \end{bmatrix} \begin{bmatrix} \Delta(\frac{E_1}{E}) \\ \Delta(\frac{\rho_C}{\rho_B}) \\ \Delta(\frac{t}{h}) \end{bmatrix} = \begin{bmatrix} \Delta_{IJ} \end{bmatrix} \begin{bmatrix} \Delta(\frac{E_1}{E}) \\ \Delta(\frac{\rho_C}{\rho_B}) \\ \Delta(\frac{t}{h}) \end{bmatrix} \tag{36}$$

However, since the uncertainty in the storage modulus ratio, E_1/E, is not directly observable, it is necessary to introduce a change in variables so that uncertainties in $\Delta E_1/E$ ΔS_1 and ΔS_2 (the new dependent variables) may be expressed in terms of uncertainties in $\Delta(\omega/\omega_0)^2$, $\Delta(\rho_C/\rho_B)$, and $\Delta(t/h)$ (the new independent variables). From the first of Eq. (36) a change in frequency at constant density and thickness ratios gives:

$$\frac{\partial(E_1/E)}{\partial(\omega/\omega_0)^2} = 1/\frac{\partial(\omega/\omega_0)^2}{\partial(E_1/E)}\Big|_{\rho,t} = 1/\Delta_{11} \tag{37}$$

From Eq. (36), a change in density, evaluated at constant frequency and thickness ratios gives:

$$\frac{\partial(E_1/E)}{\partial(\rho_C/\rho_B)} = -\frac{\partial(\omega/\omega_0)^2}{\partial(\rho_C/\rho_B)}\Big|_{E,t} / \frac{\partial(\omega/\omega_0)^2}{\partial(E_1/E)}\Big|_{\rho,t} = -\Delta_{12}/\Delta_{11} \tag{38}$$

Also from Eq. (36), a change in thickness at constant frequency and density ratios shows that:

$$\frac{\partial(E_1/E)}{\partial(t/h)} = -\frac{\partial(\omega/\omega_0)^2}{\partial(t/h)}\Big|_{E,\rho} / \frac{\partial(\omega/\omega_0)^2}{\partial(E_1/E)}\Big|_{\rho,t} = -\Delta_{13}/\Delta_{11} \tag{39}$$

With these, we may evaluate the uncertainty to be expected in the determination of the storage modulus as:

$$\begin{aligned} \Delta(E_1/E) &= \frac{\partial(E_1/E)}{\partial(\omega/\omega_o)^2}\Delta(\frac{\omega}{\omega_o})^2 + \frac{\partial(E_1/E)}{\partial(\rho_C/\rho_B)}\Delta(\frac{\rho_C}{\rho_B}) + \frac{\partial(E_1/E)}{\partial(t/h)}\Delta(\frac{t}{h}) \\ &= \left|\frac{1}{\Delta_{11}}\right|(\frac{\omega}{\omega_o})^2 f_{\omega^2} + \left|\frac{\Delta_{12}}{\Delta_{11}}\right|(\frac{\rho_C}{\rho_B}) f_\rho + \left|\frac{\Delta_{13}}{\Delta_{11}}\right|(\frac{t}{h}) f_t \end{aligned} \tag{40}$$

where the frequency, density and thickness ratios are those of the nominal configuration. Note that the total uncertainty is the sum of the absolute values of the uncertainties due to estimates of the uncertainties in the three observable quantities.

We may express the uncertainties in the stiffness ratios in terms of the same three observable quantities as:

$$\Delta(S_I) = \frac{\partial(S_I)}{\partial(\omega/\omega_o)^2}\Big|_{\rho,t}\,\Delta(\frac{\omega}{\omega_o})^2 + \frac{\partial(S_I)}{\partial(\rho_C/\rho_B)}\Big|_{\omega,t}\,\Delta(\frac{\rho_C}{\rho_B}) + \frac{\partial(S_I)}{\partial(t/h)}\Big|_{\omega,t}\,\Delta(\frac{t}{h}) \tag{41}$$

for I = 1 or 2. For a change in the stiffness ratio, $\Delta E_1/E$, at constant density and thickness, we find:

$$\frac{\partial S_I}{\partial(\omega/\omega_0)^2} = \frac{\partial S_I}{\partial(E_1/E)}\Big|_{\rho,t} / \frac{\partial(\omega/\omega_0)^2}{\partial(E_1/E)}\Big|_{\rho,t} \tag{42}$$

For a change in the density ratio at constant frequency and thickness, the second and third of Eq. (36) give:

$$\frac{\Delta S_I}{\Delta(\rho_C/\rho_B)} = \frac{\partial(S_I)}{\partial(\rho_C/\rho_B)}\Big|_{E,t} + \frac{\partial S_I}{\partial(E_1/E)}\Big|_{\rho,t}\ \frac{\partial(E_1/E)}{\partial(\rho_C/\rho_B)}\Big|_{\omega,t} \tag{43}$$

for I = 1 or 2. Similarly, using once again the second and third of Eq. (36), but now evaluated for a change in thickness at constant frequency and density, we find that:

$$\frac{\Delta S_I}{\Delta(t/h)} = \frac{\partial S_I}{\partial(t/h)}\Big|_{E,\rho} + \frac{\partial S_I}{\partial(E_1/E)}\Big|_{\rho,t}\ \frac{\partial(E_1/E)}{\partial(t/h)}\Big|_{\omega,\rho} \tag{44}$$

By using Eqs. (37-39) and Eqs. (42-44), the dependence of the uncertainties of the quantities E_1/E and the stiffness ratios S_1 and S_2 on the uncertainties in the observable frequency, density, and thickness ratios may be established in terms of the elements of the array of Eq. (36) as:

$$\begin{bmatrix} \Delta(E_1/E) \\ \Delta S_1 \\ \Delta S_2 \end{bmatrix} = \frac{1}{\Delta_{11}} \begin{bmatrix} 1 & -\Delta_{12} & -\Delta_{13} \\ \Delta_{21} & \Delta_{22}\Delta_{11} - \Delta_{21}\Delta_{12} & \Delta_{23}\Delta_{11} - \Delta_{21}\Delta_{13} \\ \Delta_{31} & \Delta_{32}\Delta_{11} - \Delta_{31}\Delta_{12} & \Delta_{33}\Delta_{11} - \Delta_{31}\Delta_{13} \end{bmatrix} \begin{bmatrix} \Delta(\omega/\omega_0)^2 \\ \Delta(\rho_C/\rho_B) \\ \Delta(t/h) \end{bmatrix} \tag{45}$$

This enables the evaluation of the uncertainty in the not-directly-observable modulus and stiffness ratios in terms of more readily observable parameters. Note that the array of Eq. (45) must to be evaluated at a reference configuration and that the first index denotes the row, and the second, the column, of the array of Eq. (36).

If we then express the estimates of the uncertainties in the observable parameters as fractions of the nominal values of the reference configuration, i.e.,

$$f_{\omega^2} = \frac{|\Delta(\omega/\omega_0)^2|}{(\omega/\omega_0)^2} \qquad f_\rho = \frac{|\Delta(\rho_C/\rho_B)|}{(\rho_C/\rho_B)} \qquad f_t = \frac{|\Delta(t/h)|}{(t/h)} \tag{46a,b,c}$$

We may now combine the uncertainty due to system parameters, Eq. (45) with the uncertainty due to the measurement of loss factors, Eq. (34) to form the total uncertainty to be expected in the determination of the loss modulus as:

$$\begin{aligned} \Delta(E_2/E) = {} & S_1\eta_{SYS} f_S + S_2\eta_{BARE} f_B + |\eta_S\{\Delta_{21}/\Delta_{11}\} - \eta_B\{\Delta_{31}/\Delta_{11}\}|(\omega/\omega_o)^2 f_{\omega^2} \\ & + |\eta_S\{\Delta_{22} - \Delta_{21}\Delta_{12}/\Delta_{11}\} - \eta_B\{\Delta_{32} - \Delta_{31}\Delta_{12}/\Delta_{11}\}|(\rho_C/\rho_B) f_\rho \\ & + |\eta_S\{\Delta_{23} - \Delta_{21}\Delta_{13}/\Delta_{11}\} - \eta_B\{\Delta_{33} - \Delta_{31}\Delta_{13}/\Delta_{11}\}|(t/h) f_t \end{aligned} \tag{47}$$

Note that as the five uncertainties are independent quantities, the absolute value of each must be taken before summing.

The array of error coefficients defined in Eq. (36) and appearing in Eq. (47) may be evaluated analytically, as is shown elsewhere [5]. Alternatively, numerical values for the quantities $(\omega/\omega_0)^2$, S_1, and S_2 may be evaluated for a nominal or reference configuration. Then, by re-evaluating after making small changes individually in each of the ratios E_1/E, ρ_C/ρ_B and t/h, numerical approximations to the values of the partial derivatives in the array of Eq. (36) may be found. Using these as the elements of the error array, together with the nominal values of parameters and the uncertainties in observed quantities, enables the evaluation of the uncertainties in loss and storage moduli with Eqs. (40 and 47).

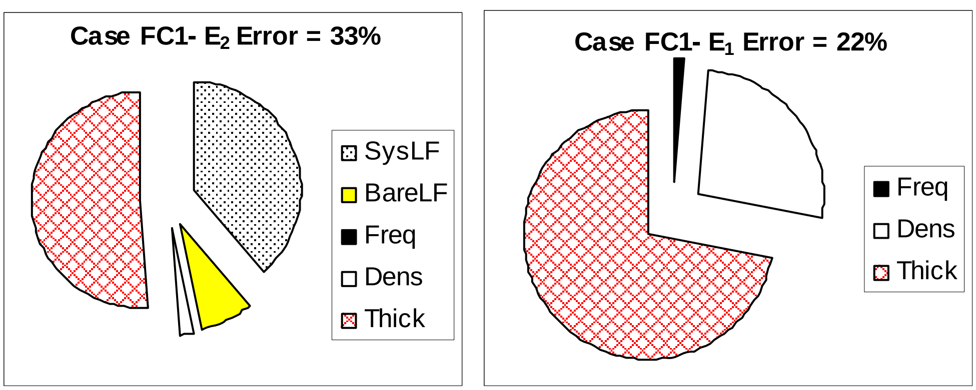
EXAMPLES

As examples of the uncertainties to be expected, we consider cantilever beams, fully and partially coated on both sides, vibrating at resonance in the 2nd bending mode. We choose a beam of length 200.9 mm, thickness 2.0 mm, density 4.372 gm/cc, and modulus 115 GPa (titanium); with a coating having density 3.445 gm/cc in all cases and a storage modulus with a nominal value of 40.0 GPa. The coating material is assumed to have a nominal amplitude independent material loss factor, chosen as representative (η_{MAT} =0.04) of the loss factor of plasma-sprayed Mg spinel for peak strains of several hundred ppm [4]. The system Q for the bare beam is assumed to be 1000.

We assume that each loss factor is measured with an uncertainty of 10% and that frequencies are measured to within 0.5% (so that the uncertainty in the squared ratio is 2%). We also assume that the beam and coating thickness are each known to within 0.0005 in (0.0127 mm); that the density of the beam is known to within 1%, and the density of the coating to within 5%. For these examples, the derivatives of Eq. (36) were evaluated numerically by evaluating the frequency and stiffness ratios before and after individually changing E_1/E, ρ_C/ρ_B, and t/h by 2%.

As Case FC1 we consider a fully covered beam with a relatively thin (0.0835 mm) coating. For the material considered, the system Q is 229. As Case FC2 we consider the same beam, but with coating thickness doubled to 0.167 mm. In this case, the system Q is 131. As Case PC1 we consider a beam with a 0.668 mm coating, 30 mm long, centered at 53% of span. In case PC2 we consider the same beam, but assume the uncertainty in determination of frequencies and loss factors can be reduced by 50%. Additional examples and a more comprehensive development of the methodology are given elsewhere [5].

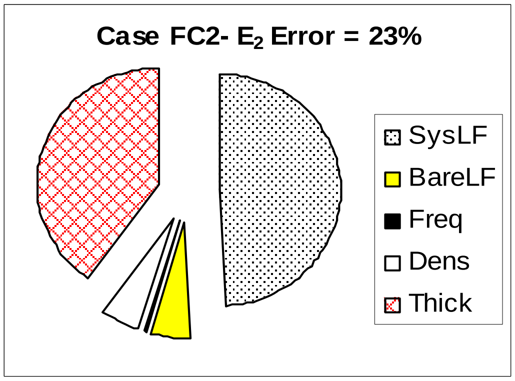
For the parameters of Case FC1, the total uncertainty in the determination of the loss modulus is found to be 33%, with the identification of sources given in Fig. 2a. The total uncertainty in the storage modulus is 22%, with the distribution as in Fig. 2b. In the case of the fully covered beam, the limiting factor in obtaining precise determinations of the modulus is clearly the uncertainty in the determination of the thickness ratio of a thin coating.

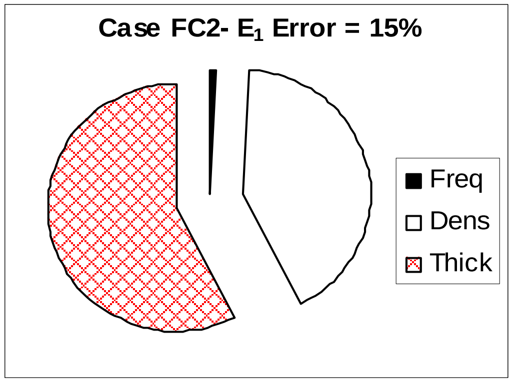


a. Loss Modulus b. Storage Modulus

Figure 2. Uncertainties in Determination of Modulii: Case FC1

As it does not seem likely that coating thicknesses can be maintained or measured to greater precision than ½ mil, the uncertainty can be reduced only by increasing the coating thickness. Thus, we consider Case FC2, with parameters the same as Case FC1, except that the thickness is doubled. This, of course, also increases the loss factor of the coated system so that. Q=131. While the errors are reduced, they still remain significant, as shown in Fig. 3. As the error due to uncertainty in coating thickness remains large, an increase in the precision of measurement of frequency and loss factors will not improve the estimate of the storage modulus. However, in this case, reducing by 50% the uncertainty in the loss factors would reduce the uncertainty in the loss modulus to about 17%.

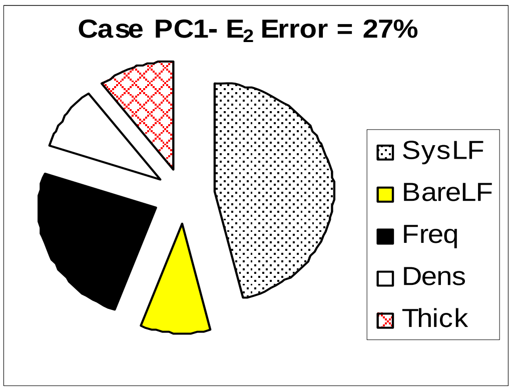


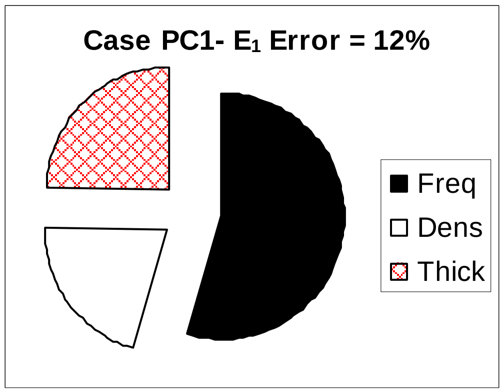


a. Loss Modulus b. Storage Modulus

Figure 3. Uncertainties in Determination of Modulii: Case FC2

For the system considered as Case PC1, the total uncertainty in the storage modulus ratio E_2/E, is 27%, with the sources of the uncertainty as identified in Fig. 4a. The total uncertainty in the storage modulus is 12%, with the sources as shown in Fig. 4b. The uncertainties in the determination of the modulii are seen to be reduced significantly from those seen with fully covered beams, primarily due to the increased certainty to which the thicker coating (4x that of case Ia) can be measured.

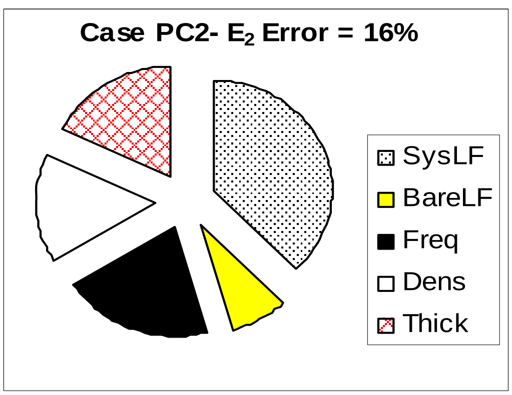


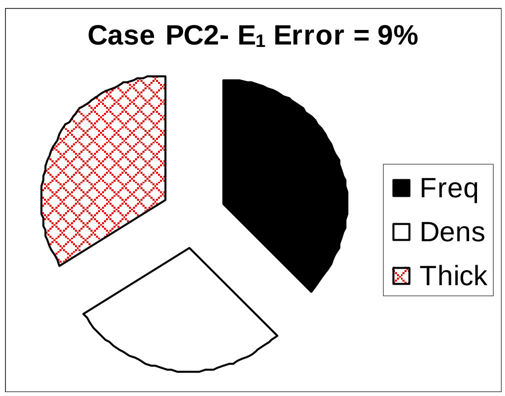


a. Loss Modulus b. Storage Modulus

Figure 4. Uncertainties in Determination of Modulii: Case PC1

If the uncertainty in frequencies and loss factors can be reduced by 50%, the errors (especially in the loss modulus) are further reduced, as is shown in Figs. 5a and 5b.





a. Loss Modulus b. Storage Modulus

Figure 5. Partially Covered Beam with Improved Measurements: Case PC2

SUMMARY AND CONCLUSIONS

The accuracy achievable in the determination of a loss modulus from experimental data on fully covered beams of typical configuration is dependent upon the accuracy to which five quantities are determined. These are: (1) the ratio of the thickness of coating and substrate, (2) the total system loss factor, (3) the ratio of densities of coating and substrate, (4) the extraneous losses (i.e. bare beam damping), and (5) the ratio of resonant frequencies of the coated and uncoated beams, For the typical configurations considered and the assumed levels of accuracy in measurement, the first two are of greatest significance, and the fifth is of the least. The determination of the storage modulus is dependent on the first, third and fifth, with the thickness ratio being the most significant for the cases considered.

In the case of the partially covered beam, the uncertainty in the measurement of the system loss factor of the coated beam is the dominant source of uncertainty in the determination of the loss modulus, with the ratio of resonant frequencies being second. For the parameters of the system considered, uncertainties in the thickness and density ratios appear to be of minor significance. In the determination of the coating storage modulus using a partially coated beam, it was found that it is the uncertainty in the determination of the ratio of the frequencies of coated and uncoated beams has the greatest impact on the accuracy to which the modulus may be estimated.

The thicker coatings required for the partially coated beam lead to a significantly lower influence of error in thickness measurement. As it was assumed that thickness measurements could be made only to a precision of 0.0005 in, and as Case PC1has 8 times the thickness of Case FC1, the proportionate error is much less.

The much higher significance of an inaccurate measurement of the frequencies of a partially coated beam is a consequence of the inherently lower strain energy ratio. The storage modulus is (essentially) determined from degree to which the frequency ratio differs from unity. When the strain energy ratio is reduced, this difference becomes small and uncertainty of measurement becomes critical. This precludes the use of significantly thinner coatings, unless frequencies can be measured with significantly better repeatability than that assumed in this work.

For tests constrained to similar, moderate, values of coated-system Q (e.g. 225), the partially coated beam appears to enable more accurate determinations of the modulii. Moreover, by avoiding the need to deduce the actual material properties from the effective values, additional sources of error are avoided.

For partially coated beams, the use of increased coating thicknesses beyond those considered here does not appear to be viable, as the strain energy ratio achieves a maximum at (about) the parameters of Case PC1, and a higher modulus or greater thickness leads to a lower strain energy ratio, and a lower system damping. For the fully covered beams, the strain energy ratio increases more rapidly than the coating thickness and thicker coatings may be used to reduce the dependence on the uncertainty in thickness while increasing the system loss factor. This also reduces the influence of the extraneous dissipation.

These results, although obtained only with experience-based estimates of uncertainties in observable quantities, suggest that uncertainties in the determination of coating modulii on the order of 20% may be expected.

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