

# **Normal Mode Theory**

**Training Seminar, 8:00-9:00 am**

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# Outline

- Objectives of NMT
- Comments on Damping
- Comments on NMT
- Orthogonality
- Modal Expansion
- Application to Base Motion
- Modal Mass
- DDAM

# Basic Objectives of Normal Mode Theory

- **Reduce the equations of motion of a multi-degree-of-freedom dynamic system to a single degree-of-freedom system on a mode by mode basis.**
- **The theory is the basis of DDAM shock analysis and seismic analyses of structures.**
- **As used in these analyses, frequencies and mode shapes are calculated without damping.**
- **DDAM assumes no damping in the Shock Spectrum inputs; seismic analyses includes damping in the Spectrum Response curves.**

# Damping

- Forces from linear viscous damping are assumed proportional to the velocity.

$$F_d = CV = C \frac{dx}{dt}$$

- Most fluid damping will vary as the velocity squared- equations are nonlinear. Coulomb is also nonlinear

$$F_d = C_d V^2$$

# More on Damping

- **Structural Damping. Energy dissipated is proportional to displacement squared.**

$$W = aX^2$$

$$C_{eq} = \frac{a}{pW}$$

- **For linear viscous damping, C is a constant.**

# More on Damping

- The linear viscous damping,  $C$  is a an approximation to the energy loss.
- DDAM assumes no damping because in many areas of a ship, the peak shock spectrum value at typical frequencies occurs early and normal damping of structures of 2%-5% is not significant.
- However on masts and antennas it is a different story – peaks occurs late in time.

# Comments on DDAM

- **Normal Mode Theory is the basis of DDAM.**
- **2D Theory - Input and Response are in the same direction.**
- **Three dimensional theory is normally used.**
- **Input in one direction and response in all DOF.**
- **No damping in DDAM Normal Mode Theory.**
- **Can add damping to the spectrum response curves.**
- **System must be linear – no gaps or nonlinear springs.**
- **Same input at all foundation points.**

# Orthogonality

- **Orthogonality is used to break down the equations of motion in a MDOF system into a SDOF in each mode of vibration.**
- **The time-history response can be calculated on a mode-by-mode basis.**
- **A shock or response spectrum is the peak response of the input motion at a particular frequency.**
- **Once a spectrum input is introduced, the time of the peak response in each mode is lost.**

# Natural Frequencies

**Equations of Motion:**

$$[M] \{\ddot{x}(t)\} + [K] \{x(t)\} = 0$$

**Assume**

$$\{x(t)\} = \{\bar{X}\} \sin \omega t$$

**Substituting**

$$\omega^2 [M] \{\bar{X}\} = [K] \{\bar{X}\}$$

# Natural Frequencies

The natural frequencies can be found in a number of ways. One is to solve the Frequency determinant.

$$\left| [M] - \frac{1}{\omega^2} [K] \right| \{ \bar{X} \} = 0$$

From this N-mass system, N natural frequencies,  $\omega_a$ , and N mode shapes or eigenvectors,  $\{ \bar{X}_a \}$ , can be found.

# Orthogonality of the Normal Modes

$$\mathbf{w}_b^2 [m] \{\bar{x}_b\} = [k] \{\bar{x}_b\}$$

$$\mathbf{w}_a^2 [m] \{\bar{x}_a\} = [k] \{\bar{x}_a\}$$

Premultiply the first equation by  $\{\bar{x}_a\}^t$

and the second equation by  $\{\bar{x}_b\}^t$

## Orthogonality Continued

$$\mathbf{W}_b^2 \left\{ \bar{x}_a \right\}^t [m] \left\{ \bar{x}_b \right\} = \left\{ \bar{x}_a \right\}^t [k] \left\{ \bar{x}_b \right\}$$

and:

$$\mathbf{W}_a^2 \left\{ \bar{x}_b \right\}^t [m] \left\{ \bar{x}_a \right\} = \left\{ \bar{x}_b \right\}^t [k] \left\{ \bar{x}_a \right\}$$

# Orthogonality Continued

$$[[A][B][C]]^t = [C]^t [B]^t [A]^t$$

**The mass and stiffness matrices are symmetric**

$$[M]^t = [M] \text{ and } [K]^t = [K]$$

**Taking the transpose of the entire first equation**

$$\mathbf{W}_b^2 \{ \bar{x}_a \}^t [m] \{ \bar{x}_b \} = \{ \bar{x}_a \}^t [k] \{ \bar{x}_b \}$$

$$\mathbf{W}_b^2 \{ \bar{x}_b \}^t [m] \{ \bar{x}_a \} = \{ \bar{x}_b \}^t [k] \{ \bar{x}_a \}$$

# Orthogonality Continued

**Thus**

$$\mathbf{w}_b^2 \{ \bar{x}_b \}^t [m] \{ \bar{x}_a \} = \{ \bar{x}_b \}^t [k] \{ \bar{x}_a \}$$

**and**

$$\mathbf{w}_a^2 \{ \bar{x}_b \}^t [m] \{ \bar{x}_a \} = \{ \bar{x}_b \}^t [k] \{ \bar{x}_a \}$$

**Subtraction yields the following**

$$(\mathbf{w}_a^2 - \mathbf{w}_b^2) \{ \bar{x}_b \}^t [m] \{ \bar{x}_a \} = 0$$

# Orthogonality Continued

Thus, for any two modes, a and b.

$$\{\bar{x}_a\}^t [M] \{\bar{x}_b\} = 0$$

$$\{\bar{x}_a\}^t [K] \{\bar{x}_b\} = 0$$

If [M] is diagonal,

$$\sum \bar{x}_{ia} \bar{x}_{ib} m_i = 0$$

# Orthogonality Continued

Consider the three mass system with  
a diagonal mass matrix:

$$\begin{Bmatrix} \bar{X}_{1a} \\ \bar{X}_{2a} \\ \bar{X}_{3a} \end{Bmatrix}^T \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \bar{X}_{1b} \\ \bar{X}_{2b} \\ \bar{X}_{3b} \end{Bmatrix} =$$

$$\{ \bar{X}_{1a} \quad \bar{X}_{2a} \quad \bar{X}_{3a} \} \begin{bmatrix} m_1 \bar{X}_{1b} \\ m_2 \bar{X}_{2b} \\ m_3 \bar{X}_{3b} \end{bmatrix} =$$

$$m_1 \bar{X}_{1b} \bar{X}_{1a} + m_2 \bar{X}_{2b} \bar{X}_{2a} + m_3 \bar{X}_{3b} \bar{X}_{3a}$$

# Orthogonality Continued

If the mass matrix is diagonal, summation notation can be used:

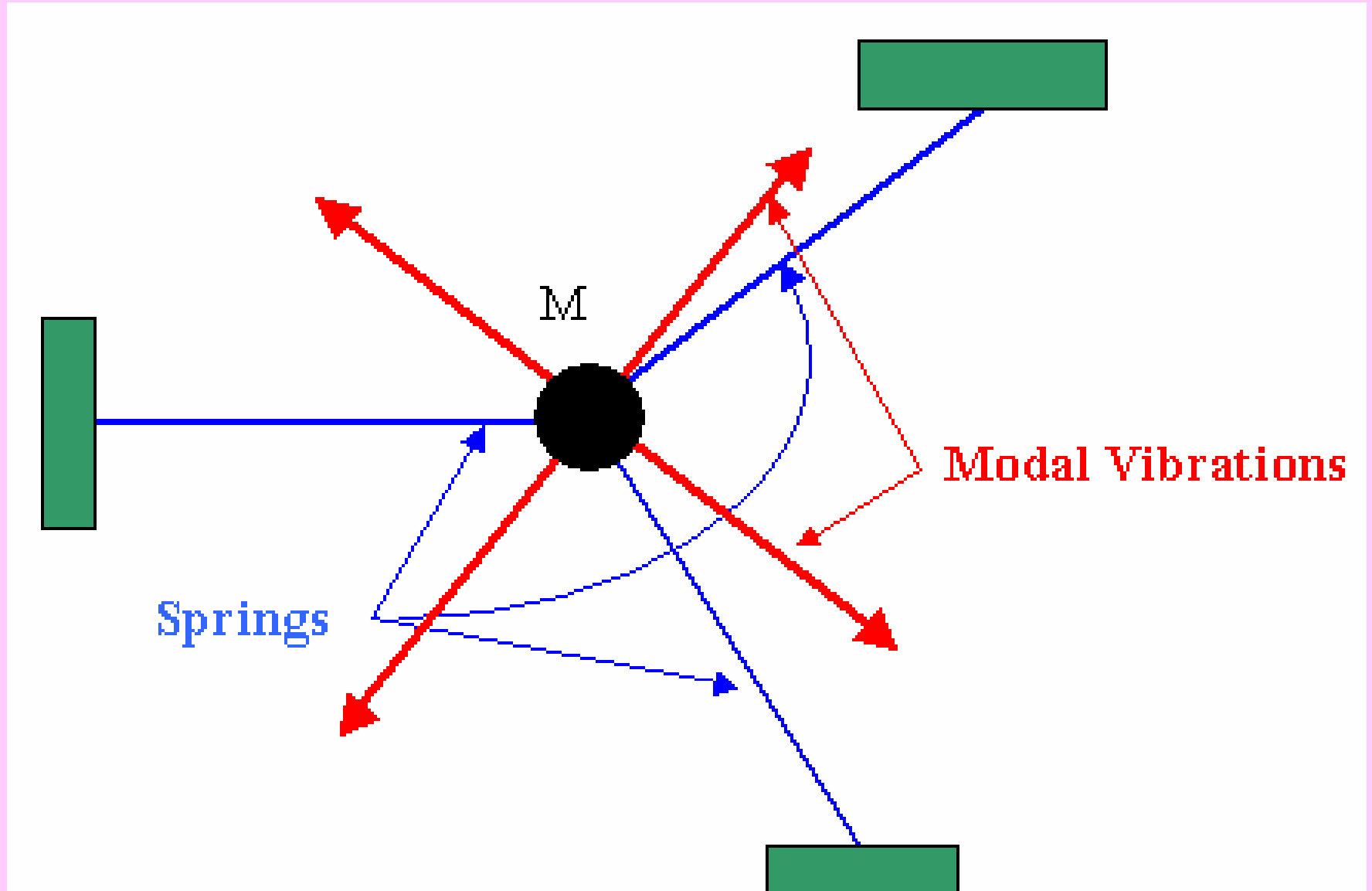
$$\sum_i \bar{X}_{ia} \bar{X}_{ib} m_i = 0$$

A useful identity is developed from

$$\mathbf{w}_a^2 [M] \{\bar{X}_a\} = [K] \{\bar{X}_a\}$$

$$\mathbf{w}_a^2 \{\bar{X}_a\}^t [M] \{\bar{X}_a\} = \{\bar{X}_a\}^t [K] \{\bar{X}_a\}$$

# Orthogonality Continued



## Orthogonality (Continued)

Mode Shape Perpendicular to Each Other

$$[M] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = m[I]$$

Thus

$$\{\bar{x}_a\}^t [M] \{\bar{x}_b\} = \{\bar{x}_a\}^t \{\bar{x}_b\} = 0$$

Two vector are orthogonal ( at right angles) if

$$\{\bar{x}_a\}^t \{\bar{x}_b\} = 0$$

# Two Usual Method to Normalize Mode Shapes

1. The Maximum value is 1.
2. The matrix product below is 1

$$\{\bar{x}_a\}^t [M] \{\bar{x}_a\} = k_a^2$$

If each element of the mode shape is divided by  $k_a$

$$\left\{ \frac{\bar{x}_a}{k_a} \right\}^t [M] \left\{ \frac{\bar{x}_a}{k_a} \right\} = 1$$

# Modal Expansion

The expansion theorem shown below is the basis of Normal Mode Theory. Basically it is stating that any motion  $\{x(t)\}$  can be expressed as the sum of the modal contributions.  $q_a(t)$  is the modal displacement, an amplifier on the mode shape that varies with time, to yield the real displacement  $\{x(t)\}$ .

$$\{x(t)\} = \sum_{a=1}^n \{\bar{X}_a\} q_a(t)$$

# More on Modal Displacements

$$x_1 = \bar{X}_{1a} q_a + \bar{X}_{1b} q_b$$

$$x_2 = \bar{X}_{2a} q_a + \bar{X}_{2b} q_b$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} \bar{X}_{1a} & \bar{X}_{1b} \\ \bar{X}_{2a} & \bar{X}_{2b} \end{bmatrix} \begin{Bmatrix} q_a \\ q_b \end{Bmatrix}$$

*or inverting the modal matrix*

$$\begin{Bmatrix} q_a(t) \\ q_b(t) \end{Bmatrix} = \begin{bmatrix} \bar{X}_{1a} & \bar{X}_{1b} \\ \bar{X}_{2a} & \bar{X}_{2b} \end{bmatrix}^{-1} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

# Response to Base Motion

The equations of motion can be written as

$$[M]\{\ddot{x}(t)\} + [K]\{x(t)\} = -[M]\{1\}\ddot{z}(t)$$

where

$$\{x(t)\} = \sum_{a=1}^n \{\bar{X}_a\} q_a(t)$$

and

$$\{\ddot{x}(t)\} = \sum_{a=1}^n \{\bar{X}_a\} \ddot{q}_a(t)$$

$$[M] \sum_{a=1}^n \{\bar{X}_a\} \ddot{q}_a(t) + [K] \sum_{a=1}^n \{\bar{X}_a\} q_a(t) = -[M]\{1\}\ddot{z}(t)$$

# Response to Base Motion

$$[M] \sum_{a=1}^n \{\bar{X}_a\} \ddot{q}_a(t) + [K] \sum_{a=1}^n \{\bar{X}_a\} q_a(t) = -[M] \{1\} \ddot{z}(t)$$

**Premultiply by a particular mode shape,  $\{\bar{X}_b\}^t$**

$$\{\bar{X}_b\}^t [M] \{\bar{X}_b\} \ddot{q}_b(t) + \{\bar{X}_b\}^t [K] \{\bar{X}_b\} q_b(t) = -\{\bar{X}_b\}^t [M] \{1\} \ddot{z}(t)$$

**Because of orthogonality, all modes but one are eliminated from the equations of motion. But,**

$$\omega_a^2 [M] \{\bar{X}_a\} = [K] \{\bar{X}_a\}$$

$$\omega_a^2 \{\bar{X}_a\}^t [M] \{\bar{X}_a\} = \{\bar{X}_a\}^t [K] \{\bar{X}_a\}$$

## Modal Participation Factor, $P_b$

$$\{\bar{X}_b\}^t [M] \{\bar{X}_b\} \ddot{q}_b(t) + \mathbf{w}_b^2 \{\bar{X}_b\}^t [M] \{\bar{X}_b\} q_b(t) = -\{\bar{X}_b\}^t [M] \{1\} \ddot{z}(t)$$

$$\ddot{q}_b(t) + \mathbf{w}_b^2 q_b(t) = -\frac{\{\bar{X}_b\}^t [M] \{1\}}{\{\bar{X}_b\}^t [M] \{\bar{X}_b\}} \ddot{z}(t)$$

The Participation Factor,  $P_b$  is defined as:

$$P_b = \frac{\{\bar{X}_b\}^t [M] \{1\}}{\{\bar{X}_b\}^t [M] \{\bar{X}_b\}}$$

$$\ddot{q}_b(t) + \mathbf{w}_b^2 q_b(t) = -P_b \ddot{z}(t)$$

# Modal Time History

$$\ddot{q}_b(t) + \mathbf{w}_b^2 q_b(t) = -P_b \ddot{z}(t)$$

**The solution to this equation of motion can be written as:**

$$q_b(t) = -\frac{P_b}{\mathbf{w}_b} \int_0^t \ddot{z}(\mathbf{t}) \sin \mathbf{w}_b(t - \mathbf{t}) d\mathbf{t}$$

**And  $\mathbf{x}(t)$**

$$\{x(t)\} = -\sum_{b=1, N} \{\bar{X}_b\} \frac{P_b}{\mathbf{w}_b} \int_0^t \ddot{z}(\mathbf{t}) \sin \mathbf{w}_b(t - \mathbf{t}) d\mathbf{t}$$

## Modal Time-History (Continued)

**Modal time-history of mass point deflections can be calculated from many FEA programs such as NASTRAN. The number of modes to be included can be specified. Of course, an eigenvalue solution must be calculated first. A direct integration of the equations of motion is usually used because it is cheaper. Forces can be determined from the stiffness matrix.**

$$\{F(t)\} = \{K\}\{x(t)\}$$

# Spectrum Inputs

$$q_b(t) = -\frac{P_b}{\omega_b} \int_0^t \ddot{z}(\tau) \sin \omega_b(t - \tau) d\tau$$

The maximum value of the integral above is the Spectrum Velocity

$$V_b = \left| \int_0^t \ddot{z}(\tau) \sin \omega_b(t - \tau) d\tau \right|_{\substack{\text{max over} \\ \text{all } t}}$$

$$q_b \Big|_{\text{max}} = \frac{P_b}{\omega_b} V_b$$

# Spectrum Inputs

The maximum physical displacements for mode **b** can be written as:

$$\{x_b\} = \{\bar{X}_b\} q_b$$

where

$$q_b \Big|_{\max} = \frac{P_b}{W_b} V_b$$

And the dynamic forces are

$$\{F_b\} = [K] \{x_b\} = [K] \{\bar{X}_b\} q_b$$

# Modal Summation Methods

**There are three common summation methods for forces, stresses displacements etc, determined from a spectrum input:**

**1. Absolute sum**  $x_i = \sum_b |x_{ib}|$

**2. SRSS**  $x_i = \sqrt{\sum_b x_{ib}^2}$

**3. NRL Sum**  $x_i = x_{ib}|_{\max} + \sqrt{\sum_b x_{ib}^2 - (x_{ib}|_{\max})^2}$

# Forces at Mass Points

$$\{F(t)\} = -[M] \sum_{b=1, N} \{\bar{X}_b\} \mathbf{w}_b P_b \int_0^t \ddot{z}(t) \sin \mathbf{w}_b (t - \mathbf{t}) d\mathbf{t}$$

**For mass “i” and mode b, the force is**

$$F_{ib}(t) = -m_i \bar{X}_{ib} \mathbf{w}_b P_b \int_0^t \ddot{z}(t) \sin \mathbf{w}_b (t - \mathbf{t}) d\mathbf{t}$$

**Using the spectrum velocity for Duhamel’s integral**

$$F_{ib} \Big|_{\max} = m_i \bar{X}_{ib} \mathbf{w}_b P_b V_b$$

# Forces at Mass Points

## Application of a G-Load, $A_b$

Since 
$$A_b = \frac{V_b \mathbf{w}_b}{g}$$

**The force at a mass point becomes:**

$$F_{ib} \Big|_{\max} = m_i \bar{X}_{ib} \mathbf{w}_b P_b V_b$$

$$F_{ib} \Big|_{\max} = W_i \bar{X}_{ib} P_b A_b$$

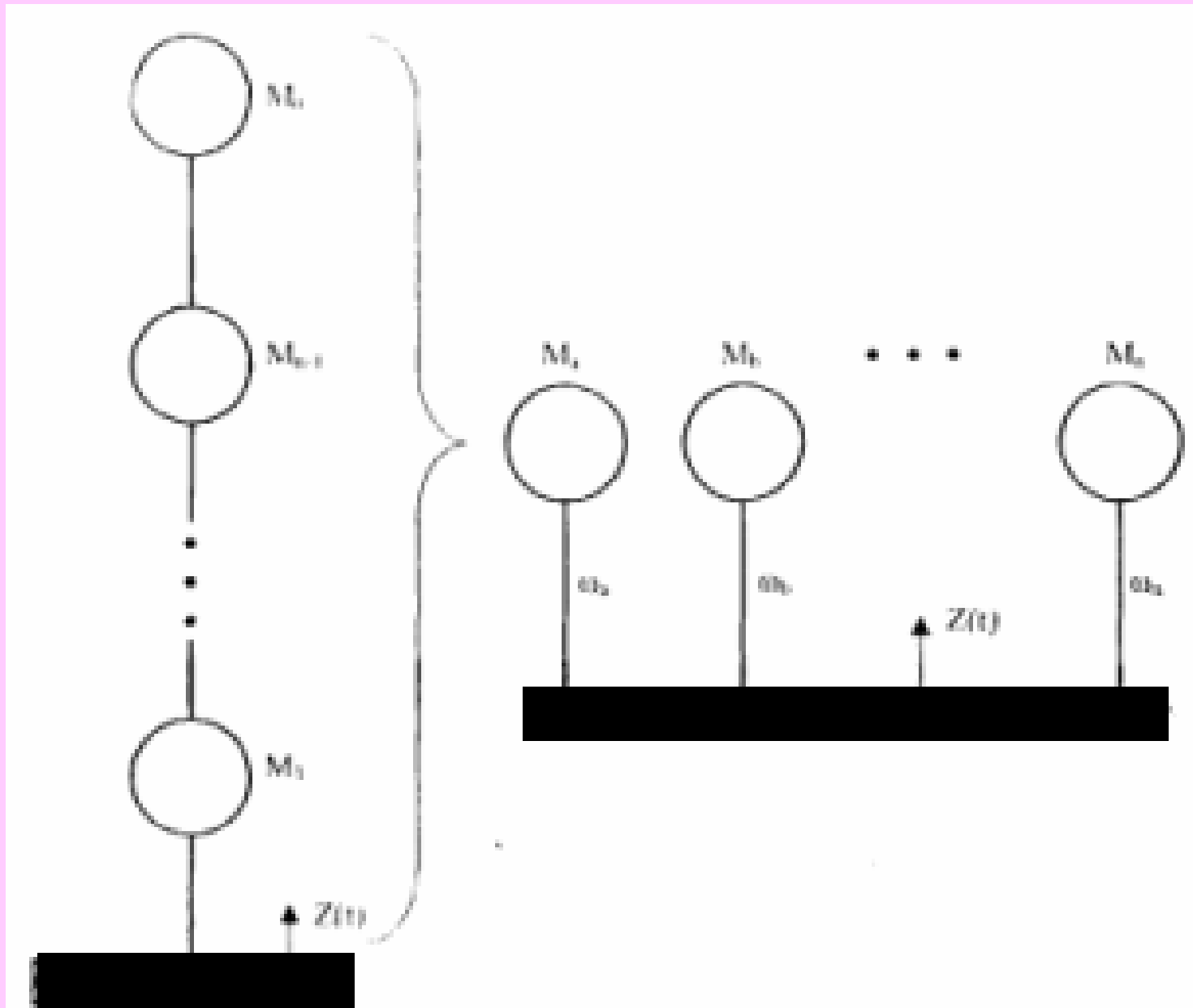
# Modal G-Load at Each Mass

$$F_{ib} \Big|_{\max} = W_i \bar{X}_{ib} P_b A_b$$

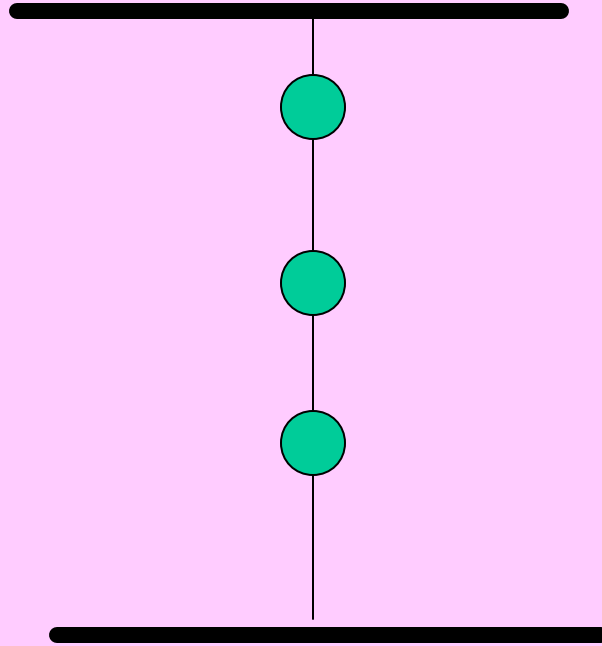
$$A_{ib} = \frac{F_{ib} \Big|_{\max}}{W_i} = \bar{X}_{ib} P_b A_b$$

**The NRL Sum is used to “combine” modes**

# Concept of a Modal Mass



# Reaction to the Foundation(s)



$$\{F(t)\} = -[M] \sum_{b=1, N} \{\bar{X}_b\} \omega_b P_b \int_0^t \ddot{z}(t) \sin \omega_b (t - \tau) d\tau$$

## Concept of a Modal Mass, (Continued)

The force acting on each mass,  $m_i$ , in each mode,  $a$ ,

$$F_{ia} = -m_i \bar{x}_{ia} \omega_a P_a \int_0^t \ddot{z}(\tau) \sin \omega_a (t - \tau) d\tau \quad (2.113)$$

Thus, by summing all forces in a mode, the total dynamic force acting on the foundation can be determined as:

$$F_a = - \sum_i F_{ia} \quad (2.114)$$

## Concept of a Modal Mass, (Continued)

$$F_o = \left( \sum_i m_i \bar{x}_{i,o} P_o \right) \omega_o \int_0^t z(\tau) \sin \omega_o (t - \tau) d\tau \quad (2.115)$$

If Eq. (2.115) is compared to the single-degree-of-freedom system

$$F = m \omega \int_0^t z(\tau) \sin \omega (t - \tau) d\tau$$

it can be seen that the terms in the parentheses have units of mass. Letting:

$$M_o = \sum_i m_i \bar{x}_{i,o} P_o$$

or:

$$M_o = \frac{\left( \sum m_i \bar{x}_{i,o} \right)^2}{\sum m_i \bar{x}_{i,o}^2}$$

# Concept of a Modal Mass, (Continued)

$$M_o = \sum_i m_i x_{io} P_o$$

$$M_o = \frac{\left(\sum m_i x_{io}\right)^2}{\sum m_i \bar{x}_{io}^2}$$

$$P_o = \frac{\sum m_i x_{io}}{\sum_i m_i \bar{x}_{io}^2}$$

If

$$\sum_i m_i \bar{x}_{io}^2 = 1$$

$$M_o = \left(\sum m_i x_{io}\right)^2 = P_o^2$$

# Generalized versus Modal Mass

$$M_a = \frac{\left[ \sum_{i=1}^n \bar{X}_{ia} m_i \right]^2}{\sum_{i=1}^n X_{ia}^2 m_i}$$

$$M_{generalized} = \sum_{i=1}^n X_{ia}^2 m_i$$

**Also, it can be shown that**

$$\sum_a M_a = \sum_i m_i = \text{total mass}$$

# Concept of a Modal Mass, (Continued)

$$M_n = \frac{\left(\sum m_i \bar{x}_{i,n}\right)^2}{\sum m_i \bar{x}_{i,n}^2}$$

the total force can be rewritten as:

$$F_n = M_n \omega_n \int_0^t z(\tau) \sin \omega_n (t - \tau) d\tau$$

Since

$$\sum_n \bar{x}_{i,n} P_n = 1$$

it can be seen that:

$$\sum_n M_n = \sum_i \sum_n m_i \bar{x}_{i,n} P_n = \sum_i m_i$$

Thus, the sum of the modal masses is equal to the sum of the structural masses.

## Concept of a Modal Mass, (Summary)

- A linear undamped dynamic system with N-masses can be presented by N SDOF systems each with a modal frequency and modal mass.
- The reaction to the foundation is identical to the original system.
- The equations above assume a diagonal mass matrix.
- In DDAM, the ONLY use for the modal mass is to determine the shock input,  $V_0$  and  $A_0$ .

# Basic Steps in DDAM

1. Calculate the natural frequencies and mode shapes.
2. Calculate the modal masses,  $M_a$ , and include enough modes so that,
$$\sum_a M_a \geq 0.80 \sum_i m_i$$
3. Determine the design spectrum inputs for each mode,  $V_0$ , and  $A_0$ .
4. Calculate the forces on each mass for each mode.

# Basic Steps in DDAM

5. Calculate the modal forces and/or stresses on each spring, beam or element.
6. Calculate the NRL sum of the forces or stresses.
7. Add the dynamic shock stresses to operational forces or stresses.
8. Compare values to allowable values.

# Comments on 3D NMT used in DDAM

1. Input in one direction can cause responses in all directions.
2. The direction of the modal participation factor and input are identical.
3. Shock spectrum inputs can be specified for the Duhamel's Integral.
4. The sum of all modal masses in the direction of the input equals the total mass. Again at least enough modes to reach 80% of the total mass must be used in each direction.
5. The effective mass in the  $r^{\text{th}}$  direction for input in the  $s$ -direction is  $M_a^{rs}$ .
6. The sum of the cross terms is zero  $\sum_a M_a^{rs} = 0$ .